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ABOUT THE STEEPNESS OF THE FUNCTION OF DISCRETE ARGUMENT

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Abstract. We introduce the notion of steepness of a coordinate-convex function of discrete argument on an ordinal-convex set. In terms of guaranteed estimates it is shown that in problems of optimization of coordinate-convex functions on an ordinal-convex set the gradient coordinatewise lifting algorithm is stable under small disturbances of the steepness of the utility function.

Keywords: steepness, gradient, algorithm, stability, discrete.

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1. Introduction

In this paper, we introduce the notion of steepness of a coordinate-convex function of discrete argument on an ordinal-convex set. In terms of guaranteed estimates, it is shown that in problems of maximization of coordinate-convex functions on an ordinal-convex set the gradient coordinatewise lifting algorithm is stable under small perturbations of the steepness of the utility function. As corollaries, we obtain improved guaranteed estimates for accuracy of the gradient algorithm, and also new sufficient conditions for the values of the utility function of the problem under consideration coincide in the global and gradient extrema.

It is needed to note that the notion of steepness was initially introduced for submodular functions and used in [1] to improve an estimate for accuracy of the gradient algorithm.

2. Definitions and notations

Let $Z_+^n(R_+^n)$ be set of $n-$dimensional non-negative integer-valued (real) vectors. $P \subseteq Z_+^n$ is a finite ordinal-convex set with zero (see, for example, [3, 4]). We introduce the following notation:

$$N(x,y) = \{i : x = (x_1, ..., x_n) \prec (y_1, ..., y_n) = y,\ x_i \prec y_i,\ x_i \neq y_i,\ 1 \leq i \leq n\},$$
$$h(x,y) = \sum_{i \in N(x,y)} h(x_i, y_i),$$
$$h(x_i, y_i) = \left|\{z_i : x_i \prec z_i \prec y_i\}\right| - 1,\ 1 \leq i \leq n,$$
$$h(x) = h(0,x),$$
$$h = h(P) = \max\{h(0,x) :\ x \in P\ \},\ r = \min\{h(x) - 1 : x \in Z_+^n \backslash P\},$$
$$fes(x,P) = \{1 \leq i \leq n : \pi_i^+(x) \in P,\ x \in P\},$$
$$\pi_i^+(x) = (x_1, ..., x_{i-1}, x_i^+ + 1, x_{i+1}, ..., x_n),$$
$$r = r(P) = \min\{h(x) - 1 \,\big|x \in Z_+^n \backslash P\}.$$

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Following [2-4], for a function : $Z_+^n \to R$ (R is the set of real numbers) we introduce the notion of the i- gradient

$$\Delta_i f(x) = f(\pi_i(x)) - f(x),$$

and the (i,j) – gradient

$$\Delta_{ij} f(x) = \Delta_j f(\pi_i(x)) - \Delta_j f(x).$$

Let $\rho = (\rho_1, ..., \rho_n) \in R_+^n$ and $\Re_\rho(Z_+^n)$ be the class of ρ-coordinate-convex functions on Z_+^n (see [2, 4]), that is, of functions $f : Z_+^n \to R$ such that for any $x \in Z_+^n$

$$\begin{array}{l} \Delta_{ij} f(x) \le 0 \quad, \; i \ne j, \quad 1 \le i, j \le n, \\ \Delta_{ii} f(x) \le -\rho_i \;, \; 1 \le i \le n. \end{array}$$

Definition 2.1. *([4]) Let $f(x)$be a non-decreasing ρ-coordinate-convex function on the set $P \subseteq Z_+^n$. The steepness of the function $f(x)$ on the set $P \subseteq Z_+^n$ is*

$$c = c(f) = \min\left\{ \frac{\rho_i}{\Delta_i f(0)} \; : \; \Delta_i f(0) > 0, \; i \in fes(0, P) \quad \right\}, \; if \; I^+(f) \ne \emptyset$$

and

$$c = c(f) = 1, \; if \;\; I^+(f) = \emptyset,$$

where

$$I^+ = I^+(f) = \{i : \Delta_i f(\pi_i(0)) \ge 0, \; i \in fes(0, P)\}.$$

We observe that if $f(x)$ is a non-decreasing ρ-coordinate-convex function on Z_+^n, then always $I^+(f) \ne \emptyset$. However, if $f(x)$ is a non-decreasing function on $P \subseteq Z_+^n$, then it is possible that $I^+(f) = \emptyset$. Indeed, let

$$f(x) = x_1 - \frac{x_1^2}{2} + x_2 - \frac{x_2^2}{2}, \quad P = \{(0,0), (1,0), (0,1)\}.$$

It is obvious that the function $f(x)$ does not decrease on the set $P \subseteq Z_+^n$ and

$$\begin{array}{l} \Delta_1 f(x) = -x_1 + 1/2, \; \Delta_2 f(x) = -x_2 + 1/2, \; \Delta_{11} f(x) = \Delta_{22} f(x) = -1, \\ \Delta_1 f(0) = 1/2 > 0, \; \Delta_2 f(0) = 1/2 > 0, \\ \Delta_1 f(\pi_1(0)) = \Delta_1 f(1,0) = -1/2, \; \Delta_2 f(\pi_2(0)) = \Delta_2 f(0,1) = -1/2, \end{array}$$

that is $f(x) \in \Re_{(1,1)}(Z_+^2)$ and $I^+ = \emptyset$.

Moreover, we observe that $0 \le c \le 1$. The inequality $c \ge 0$ is obvious. If $I^+(f) \ne \emptyset$ (the case $I^+(f) = \emptyset$ is obvious), then, because the function $f(x) \in \Re_\rho(Z_+^n)$ does not decrease on the set $P \subseteq Z_+^n$,

$$\Delta_i f(0) \ge \Delta_i f(\pi_i(0)) + \rho_i \ge \rho_i,$$

for any $i \in fes(0, P)$, that is, $c \le 1$.

We also observe that if $I^+(f) = \emptyset$ (that is, $c = 1$), then in view of nondecreasing of the function $f(x) \in \Re_\rho(Z_+^n)$ on the set $P \subseteq Z_+^n$, the equality

$$\max\{f(x) : \; x \in P\} = \max\{f(\pi_i(0)) : \; i \in fes(0, P)\}.$$

Therefore, without loss of generality, we may assume that $I^+(f) \ne \emptyset$. However, for completeness of presentation and for correctness of the definition of steepness it is needed to add the case $I^+(f) = \emptyset$, which is not a hard restriction.

3. The problem

We consider the following convex discrete optimization problem A: find

$$\max\{f(x) : x = (x_1, ..., x_n) \in P \subseteq Z_+^n\},$$

where $f(x) \in \Re_\rho(Z_+^n)$, $f(x)$ is a nondecreasing function on the set P, P is an ordinal-convex set.

Let $x^* = (x_1^*, ..., x_n^*)$ be an optimal solution of the problem A, that is, $f(x^*) = \max\{f(x) : x \in P\}$.

Let $x^g = (x_1^g, ..., x_n^g)$ be the gradient solution (the gradient maximum of the function $f(x)$ on the set P) of the problem A [2-5].

The guaranteed (relative) estimate for the error of the gradient algorithm solving the problem A is, as usual, a number $\varepsilon \geq 0$ such that

$$\frac{f(x^*) - f(x^g)}{f(x^*) - f(0)} \leq \varepsilon.$$

The disturbed problem A^δconsist of the following: find

$$\max\{f^\delta(x) \mid \ x \in P \subseteq Z_+^n\},$$

where $f^\delta(x) \in \Re_q(Z_+^n)$ is a nondecreasing function on the set P, $c(f^\delta) = c(f) + \delta$, $\delta \in R_+^1$, P is an ordinal-convex set The class of problems of type A^δ is not empty . Indeed, consider the following problem A_1^δ : find

$$\max\{f^\delta(x) = f(x) + g(x) \mid x \in P \subseteq Z_+^n \},$$

where $f^\delta(x)$, $-g(x)$ are nondecreasing functions on the set P, $I^+(f^\delta) \neq \emptyset$, $f(x) \in \Re_\rho(Z_+^n)$, $g(x) \in \Re_\mu(Z_+^n)$.

It is obvious that $f^\delta(x) \in \Re_{\rho+\mu}(Z_+^n)$. Therefore, because $\Re_{\rho+\mu}(Z_+^n) \subseteq \Re_\rho(Z_+^n)$, we find that

$$f^\delta(x) \in \Re_\rho(Z_+^n). \tag{1}$$

Since the functions $f^\delta(x)$ and $-g(x)$ do not decrease on the set$P$, we obtain

$$0 \leq \Delta_i f^\delta(x) = \Delta_i f(x) + \Delta_i g(x) \leq \Delta_i f(x), \quad \forall i \in fes(x, P), \tag{2}$$

for all $i \in fes(x, P)$, that is, $f(x)$ is a nondecreasing function on the set P. The equality

$$\Delta_i f^\delta(0) \leq \Delta_i f(0),\ \forall i \in fes(0, P) \tag{3}$$

follows from (2) for all $i \in fes(x, P)$. Since $I^+(f^\delta) \neq \emptyset$, (2) implies also that

$$0 \leq \Delta_i f^\delta(\pi_i(0)) \leq \Delta_i f(\pi_i(0))$$

that is,

$$I^+(f) \neq \emptyset. \tag{4}$$

Moreover, by virtue of (3)

$$\{i : \Delta_i f^\delta(0) > 0,\ i \in fes(0, P)\} \subseteq \{i : \Delta_i f(0) > 0,\ i \in fes(0, P)\}. \tag{5}$$

Therefore,

$$c(f^\delta) = \min\left\{ \frac{\rho_i}{\Delta_i f^\delta(0)} : \Delta_i f^\delta(0) > 0,\ i \in fes(0, P) \right\} \geq$$

$$\min\left\{ \frac{\rho_i}{\Delta_i f(0)} : \Delta_i f(0) > 0,\ i \in fes(0, P) \right\} = c(f).$$

That is, there exists $\delta \in R^1_+$ such that

$$c(f^\delta) = c(f) + \delta.$$

In particular, if in the problem A_1^δ the function $g(x)$ is of the form $g(x) = -(b, x)$, where $b = (b_1, ..., b_n) \in R^n_+$, $x = (x_1, ..., x_n) \in Z^n_+$, (b, x) is the scalar product of the vectors b and x (it is obvious that in this case relations (1)-(5) hold true), then we arrive at another class of problems of type A^δ. Taking into account relations (1)-(5), we formulate the following properties of the steepness.

Let $f(x)$ be a nondecreasing ρ-coordinate-convex function on the set $P \subseteq Z^n_+$. Then

(1) $c(af) = c(f)$ for all $a \in R^1_+$;
(2) $c(f + g) \le c(f)$, where $g(x) \in \Re_q(Z^n_+)$is a nondecreasing function on the set $P \subseteq Z^n_+, I^+(g) \neq \emptyset$;
(3) $c(f + \varphi) \ge c(f)$, where $\varphi(x) \in \Re_\beta(Z^n_+), -\varphi(x)$is a nondecreasing function on the set $P \subseteq Z^n_+, I^+(g) \neq \emptyset$;

The validity of property 1 follows from the relation

$$af(x) \in \Re_{a\rho}(Z^n_+),\ a\rho = (a\rho_1, ..., a\rho_n),\ \Delta_i af(0) = a\Delta_i f(0)$$

and the definition of the steepness.

Property 3 immediately follows from relations (1)-(5) applied to the function $f(x) + \varphi(x)$.

Property 2 is proved in the same way as property 3.

Thus, these properties allow us, in a sence, to construct perturbed problems.

In essence, stability of the gradient algorithm in terms of the guaranteed estimates means separation of the class of problems for which small disturbances of the parameters of the problem (in particular, the steepness of the utility function) do not impair the guaranteed estimates for the perturbed problems.

The prime objective of this paper is to prove the stability of the gradient coordinatewise lifting algorithm foe the problem A in terms of steepness of the utility function.

Thus, these properties allow us, in a sence, to construct perturbed problems.

Let $\varepsilon(\delta)$ and ε be guaranteed estimates for the perturbed problem $A^\delta$and the initial problem A respectively. We call the gradient coordinatewise lifting algorithm stable [3] for the problem A if

$$\varepsilon(\delta) \le K(\delta)\varepsilon,$$

where $K(\delta) \to 1$ as $\delta \to 0$.

The prime objective of this paper is to prove the stability of the gradient coordinatewise lifting algorithm foe the problem in terms of steepness of the utility function A.

4. The main result

The following theorem is the main result of this paper.

Theorem 4.1. *The gradient coordinatewise lifting algorithm is stable in the problem A under small disturbances of the steepness of the utility function.*

In order to prove the theorem, we need a series of lemmas.

Lemma 4.1. *([6]). If $f(x) \in \Re_\rho(Z^n_+)$ is a nondecreasing function, then the inequality*

$$f(y) - f(x) \le \sum_{i \in N(x,y)} h(x_i, y_i)\Delta_i f(x) - \frac{1}{2} \sum_{i \in N(x,y)} \rho_i h(x_i, y_i)(h(x_i, y_i) - 1),$$

holds true for all

$$\forall x \leq y, x, y \in Z_+^n.$$

Lemma 4.2. *([4]) Let $f(x) \in \Re_\rho(Z_+^n)$ be a nondecreasing function on the set P. Then for any $x \in P$ and $i \in fes(x, P)$*

$$\Delta_i f(\pi_i(x)) \leq (1-c)\Delta_i f(x), \ \ where \quad c = c(f).$$

Lemma 4.3. *([4]) Let $f(x) \in \Re_\rho(Z_+^n)$ be a nondecreasing on the set P function.Then the global maximum x^* and the gradient maximum x^g of the function $f(x)$ on the set $P \subseteq Z_+^n$ obey the relation*

$$\frac{f(x^*) - f(x^g)}{f(x^*) - f(0)} \leq B(c, h, r) = B,$$

where

$$B(c, h, r) = \left(\quad 1 - \frac{1}{1 + (1-c)(h-1)} \quad \right)^r.$$

Proof. Let ε and $\varepsilon(\delta)$ be the guaranteed estimates for problems and A^δ respectively. Then it follows from Lemma 4.3 that

$$\varepsilon = B(c(f), h, r), \ \varepsilon(\delta) = B(c(f^\delta), h, r).$$

Therefore, in order to prove the theorem it suffices to show that

$$B(c(f), h, r) \leq B(c(f^\delta), h, r).$$

Let δ denote the perturbation of the function $f(x)$. Then from the inequalities

$$1 + (1 - c(f) - \delta)(h-1) \leq 1 + (1 - c(f))(h-1),$$
$$\frac{1}{1 + (1 - c(f) - \delta)(h-1)} \geq \frac{1}{1 + (1 - c(f))(h-1)},$$
$$B(c(f) + \delta, h, r) = \left(\quad 1 - \tfrac{1}{1+(1-c(f)-\delta)(h-1)} \quad \right)^r \leq$$
$$\left(\quad 1 - \tfrac{1}{1+(1-c(f))(h-1)} \quad \right)^r = B(c(f), h, r)$$

We obtain

$$\varepsilon(\delta) = B(c(f^\delta), h, r) = B(c(f) + \delta, h, r) \leq B(c(f), h, r) = \varepsilon$$

□

5. Corollary and examples

Corollary 5.1. *Under the conditions of Lemma 4.3, if $c = c(f)$, then $f(x^*) = f(x^g)$.*

Let us that there exist functions in the class $\Re_\rho(Z_+^n)$ such that $c = 1$. Indeed, let

$$\max\{f(x) = -2x_1^2 + 6x_1 - x_2^2 + 3x_2 \,|\, x = (x_1, x_2) \in P\},$$

$$P = \{(0,0), (1,0), (0,1), (0,2)\}.$$

Then

$$\Delta_1 f(x) = -4x_1 + 4, \ \Delta_2 f(x) = -2x_2 + 2, \rho = (4, 2),$$
$$\Delta_1 f(0,0) = 4, \Delta_2 f(0,0) = 2.$$

Therefore,

$$c = 1.$$

Let us construct an example where $f(x^*) = f(x^g)$ *but* $c \neq 1$

That is, what Corollary 5.1 assests is sufficient but not necessary. Let

$$\max\{f(x) = -2x_1^2 + 7x_1 - x_2^2 + 3x_2 \,|\, x = (x_1, x_2) \in P\},$$

$$P = \{(0,0), (1,0), (0,1), (0,2)\}.$$

Then

$$\Delta_1 f(x) = -4x_1 + 5,\ \Delta_2 f(x) = -2x_2 + 2, \rho = (4, 2),$$
$$\Delta_1 f(0,0) = 5, \Delta_2 f(0,0) = 2, x^* = x^g = (1,0), f(x^*) = f(^g) = 5.$$

On the other hand,

$$c(f) = \min\{\rho_1/\Delta_1 f(0), \rho_2/\Delta_2 f(0)\} = \min\{4/5, 2/2\} = 4/5 \neq 1.$$

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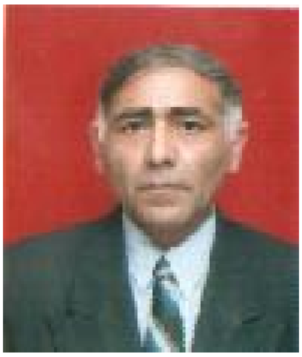

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